ON SOME CLASS OF DIFFERENTIAL-DIFFERENCE EQUATIONS ADMITTING LAX REPRESENTATION

ANDREI K. SVININ

Abstract. This note is designed to show some classes of differential-difference equations admitting Lax representation which generalize evolutionary equations known in the literature.

1. Introduction

The simplest case of an evolutionary equation sharing the property of having Lax pair representation is given by the Volterra lattice

$$r_i' = r_i(r_{i-1} - r_{i+1}) \tag{1.1}$$

which is related via the substitution $r_i = u_i u_{i+1}$ to its modified version

$$u_i' = u_i^2(u_{i-1} - u_{i+1}). \tag{1.2}$$

In turn, the substitution

$$r_i = \frac{1}{(v_i - v_{i-2})(v_{i+1} - v_{i-1})}$$

gives the relationship of the Volterra lattice (1.1) with one of an equation of Volterra type [9]

$$v_i' = \frac{1}{v_{i+1} - v_{i-1}}. \tag{1.3}$$

Remark that all the equations (1.1), (1.2) and (1.3) are evolutionary ones. Moreover these equations admit corresponding hierarchies of generalized symmetries which can be presented in explicit form via some discrete polynomials [7], [8]. It is common of knowledge that equations (1.1), (1.2) and (1.3) have corresponding integrable generalizations. For example, the equation

$$r_i' = r_i\left(\sum_{j=1}^{n} r_{i-j} - \sum_{j=1}^{n} r_{i+j}\right),$$

known as Itoh-Narita-Bogoyavlenskii lattice [3], [4], [2], naturally generalizes the Volterra lattice (1.1), while its modified version looks as [2]

$$u_i' = u_i^2\left(\prod_{j=1}^{n} u_{i-j} - \prod_{j=1}^{n} u_{i+j}\right).$$

Date: April 30, 2013.

Corresponding integrable generalization of equation (1.3), namely,

$$v_i' = \frac{1}{\prod_{j=1}^{n}(v_{i-j+n+1} - v_{i-j})}$$

and explicit form of its hierarchy was given in [8]. Our main goal of this note is to present further generalization of these equations.

In section 2, we consider some class of auxiliary linear equations and look for compatibility conditions for these ones. In a result, we get differential-difference equations in a sense admitting Lax pair representation. It is worth remarking that resulting equations, generally speaking, are not of evolutionary type. In section 3, we consider Darboux transformation of auxiliary linear equations and derive some class of quadratic discrete equations as a condition of compatibility of corresponding linear discrete equations. We observe that obtained equations in a sense generalize known in the literature the lattice potential KdV equation.

2. Linear equations and its consistency conditions

2.1. **The first class of differential-difference equations.** Let us consider the following pair of linear equations:

$$zs_i\phi_{i+n} + \phi_i = z\phi_{i+p}, \quad \phi_i' = z\xi_i\phi_{i+n}, \tag{2.1}$$

on some wave function $\phi = \phi_i = \phi_i(x, z)$. By assumption, they are parameterized by some integers $p > n \geq 1$. As is seen, these equations constitute compatible pair provided that two relations, namely,

$$s_i\xi_{i+n} = s_{i+n}\xi_{i+p}, \quad s_i' = \xi_{i+p} - \xi_i \tag{2.2}$$

are valid. Remark that the first relation in (2.2) can be equivalently rewritten as

$$\prod_{j=1}^{p-n}\xi_{i+j-1}\prod_{j=1}^{n}s_{i-j} = \delta \tag{2.3}$$

with some arbitrary constant δ. By suitable reparameterization, we can make $\delta = 1$. It is a simple observation that putting

$$\xi_i = \prod_{j=1}^{n}u_{i+j+n-1}, \quad s_i = \prod_{j=1}^{p-n}\frac{1}{u_{i+j+2n-1}}, \tag{2.4}$$

with some field $u = u_i$, we solve (2.3) with $\delta = 1$. This ansatz, after substituting it in the second relation in (2.2), gives differential-difference equation

$$\left(\prod_{j=1}^{p-n}\frac{1}{u_{i+j-1}}\right)' = \prod_{j=1}^{n}u_{i-j+p} - \prod_{j=1}^{n}u_{i-j} \tag{2.5}$$

which we can rewrite as

$$\sum_{j=1}^{p-n}\frac{u_{i+j-1}'}{u_{i+j-1}} = \prod_{j=1}^{p}u_{i+j-n-1} - \prod_{j=1}^{p}u_{i+j-1} \tag{2.6}$$

or in the form

$$\begin{aligned}\left(\prod_{j=1}^{p-n} u_{i+j-1}\right)' &= \prod_{j=1}^{p-n} u_{i+j-1}\left(\prod_{j=1}^{p} u_{i+j-n-1} - \prod_{j=1}^{p} u_{i+j-1}\right)\\ &= \prod_{j=1}^{p-n} u_{i+j-1}^2\left(\prod_{j=1}^{n} u_{i-j} - \prod_{j=1}^{n} u_{i+j+p-n-1}\right).\end{aligned} \tag{2.7}$$

Relations (2.5) and (2.6) can be considered as local differential-difference conservation laws for equation (2.7).

2.2. **The second class of differential-difference equations.** Let us introduce the potential v_i by $s_i = v_{i+p} - v_i$. The relationship between two fields u and v, due to (2.4), is given by relations

$$v_{i+p} - v_i = \prod_{j=1}^{p-n} \frac{1}{u_{i+j+2n-1}}, \quad v_i' = \xi_i = \prod_{j=1}^{n} u_{i+j+n-1}.$$

Therefore the first relation in (2.2) becomes

$$(v_{i+p-n} - v_{i-n})\, v_i' = (v_{i+p} - v_i)\, v_{i+p-n}'. \tag{2.8}$$

The latter in fact is equivalent to the differential-difference equation

$$\prod_{j=1}^{p-n} v_{i+j-1}' \cdot \prod_{j=1}^{n} (v_{i-j+p} - v_{i-j}) = 1 \tag{2.9}$$

Thus, we have in hand two classes of nonlinear differential-difference equations, namely, (2.7) and (2.9) which gives the compatibility of the linear equations (2.1). These equations involve two integers $p > n$ and $n \geq 1$, but one can see that to separate really different equations, one must suppose that p and n are co-prime positive integers. For example, this is the case $p = n + 1$, for $n \geq 1$.

2.3. **The third class of differential-difference equations.** Let

$$r_i \equiv \frac{\xi_{i-n}}{s_{i-n}} = \frac{\xi_{i+p-2n}}{s_{i-2n}} \tag{2.10}$$

$$= \frac{v_{i-n}'}{s_{i-n}} = \frac{v_{i+p-2n}'}{s_{i-2n}} \tag{2.11}$$

$$= \prod_{j=1}^{p} u_{i+j-1}. \tag{2.12}$$

In virtue of (2.9) and (2.11),

$$\prod_{j=1}^{p-n} r_{i+j-1} = \prod_{j=1}^{p} \frac{1}{s_{i+j-2n-1}} = \prod_{j=1}^{p} \frac{1}{(v_{i+j+p-2n-1} - v_{i+j-2n-1})}.$$

Then

$$\begin{aligned}\left(\prod_{j=1}^{p-n} r_{i+j-1}\right)' &= -\prod_{j=1}^{p}\frac{1}{s_{i+j-2n-1}}\cdot\sum_{j=1}^{p}\frac{s'_{i+j-2n-1}}{s_{i+j-2n-1}} \\ &= \prod_{j=1}^{p-n} r_{i+j-1}\left(\sum_{j=1}^{p}\frac{v'_{i+j-2n-1}}{s_{i+j-2n-1}}-\sum_{j=1}^{p}\frac{v'_{i+j+p-2n-1}}{s_{i+j-2n-1}}\right).\end{aligned}$$

Using (2.11) we get

$$\begin{aligned}\left(\prod_{j=1}^{p-n} r_{i+j-1}\right)' &= \prod_{j=1}^{p-n} r_{i+j-1}\left(\sum_{j=1}^{p} r_{i+j-n-1}-\sum_{j=1}^{p} r_{i+j-1}\right) \\ &= \prod_{j=1}^{p-n} r_{i+j-1}\left(\sum_{j=1}^{n} r_{i-j}-\sum_{j=1}^{n} r_{i+j+p-n-1}\right).\end{aligned} \tag{2.13}$$

One can see that (2.12) relates two equations (2.13) and (2.7).

2.4. **Linear equations on new wave function.** Let us introduce γ_i such that $\gamma_{i+1} = \gamma_i/u_{i+n}$, then

$$\gamma_{i+p} = \gamma_i\cdot\prod_{j=1}^{p}\frac{1}{u_{i+j+n-1}} = \frac{s_i}{\xi_i}\gamma_i = \frac{\gamma_i}{r_{i+n}},$$

$$\gamma_{i+p-n} = \gamma_i\cdot\prod_{j=1}^{p-n}\frac{1}{u_{i+j+n-1}} = s_{i-n}\gamma_i \tag{2.14}$$

and

$$\gamma_{i+n} = \gamma_i\cdot\prod_{j=1}^{n}\frac{1}{u_{i+j+n-1}} = \frac{\gamma_i}{\xi_i}.$$

Let $\phi_i \equiv \gamma_i\psi_i$. The linear equations (2.1) in terms of new wave function $\psi = \psi_i$ become

$$z\psi_{i+n} + r_{i+n}\psi_i = z\psi_{i+p}, \quad \psi_i' = z\psi_{i+n} - \frac{\gamma_i'}{\gamma_i}\psi_i. \tag{2.15}$$

With (2.14) and the second equation in (2.2), we have the following:

$$\left(\frac{\gamma_{i+p-n}}{\gamma_i}\right)' = \frac{\gamma_{i+p-n}}{\gamma_i}\left(\frac{\gamma'_{i+p-n}}{\gamma_{i+p-n}}-\frac{\gamma_i'}{\gamma_i}\right) = s'_{i-n} = \xi_{i+p-n}-\xi_{i-n}$$

and then taking into account (2.10) we get

$$\begin{aligned}\frac{\gamma'_{i+p-n}}{\gamma_{i+p-n}}-\frac{\gamma_i'}{\gamma_i} &= \frac{\xi_{i+p-n}-\xi_{i-n}}{s_{i-n}} = \frac{\xi_{i+p-n}}{s_{i-n}}-\frac{\xi_{i+p-2n}}{s_{i-2n}} \\ &= r_{i+n}-r_i.\end{aligned}$$

We can resolve the latter as

$$\sum_{j=1}^{p-n}\frac{\gamma'_{i+j-1}}{\gamma_{i+j-1}} = \sum_{j=1}^{n} r_{i+j-1}. \tag{2.16}$$

In turn, (2.16) becomes an identity if we substitute

$$\frac{\gamma_i'}{\gamma_i} = \sum_{j=1}^{n} a_{i+j-1}, \quad r_i = \sum_{j=1}^{p-n} a_{i+j-1},$$

with some field $a = a_i$. Then linear equations (2.15) become

$$z\psi_{i+n} + a_1^{[p-n]}(i+n)\psi_i = z\psi_{i+p}, \tag{2.17}$$

and

$$\psi_i' = z\psi_{i+n} - a_1^{[n]}(i)\psi_i, \tag{2.18}$$

where, by definition, $a_1^{[r]}(i) = \sum_{j=1}^{r} a_{i+j-1}$, for any integer $r \geq 1$. The consistency conditions for linear equations (2.17) and (2.18) yield a differential-difference equation

$$\sum_{j=1}^{p-n} a_{i+j-1}' = \sum_{j=1}^{p-n} a_{i+j-1} \left(\sum_{j=1}^{n} a_{i-j} - \sum_{j=1}^{n} a_{i+j+p-n-1} \right). \tag{2.19}$$

By direct calculations, one can check that the ansatz $r_i = \sum_{j=1}^{p-n} a_{i+j-1}$, relates (2.19) to (2.13).

3. Darboux transformation

3.1. **Quadratic discrete equation.** Let us discuss Darboux transformation for linear equations (2.1). We consider linear transformation in the form

$$\bar{\phi}_i = \phi_{i+p-n} + g_i\phi_i \tag{3.1}$$

with some coefficient g_i to be defined by condition that (3.1) should be Darboux transformation for (2.1). Consider the transformation of the first equation in (2.1). We have

$$\begin{aligned} z\bar{s}_i\left(\phi_{i+p} + g_{i+n}\phi_{i+n}\right) + \phi_{i+p-n} + g_i\phi_i &= z\left(\phi_{i+2p-n} + g_{i+p}\phi_{i+p}\right) \\ &= zs_{i+p-n}\phi_{i+p} + \phi_{i+p-n} + zg_{i+p}\phi_{i+p} \end{aligned}$$

and therefore

$$z\left(\bar{s}_i - s_{i+p-n} - g_{i+p}\right)\phi_{i+p} + g_i\phi_i + z\bar{s}_i g_{i+n}\phi_{i+n} = 0.$$

Requiring that (3.1) to be Darboux transformation gives the relations

$$g_{i+p} - g_i = \bar{s}_i - s_{i+p-n} = \bar{v}_{i+p} - \bar{v}_i + v_{i+p-n} - v_{i+2p-n}, \quad g_i s_i = \bar{s}_i g_{i+n}$$

the first of which is solved by $g_i = \bar{v}_i - v_{i+p-n}$ and therefore the second one is equivalent to the following discrete equation:

$$\left(\bar{v}_i - v_{i+p-n}\right)\left(v_{i+p} - v_i\right) = \left(\bar{v}_{i+p} - \bar{v}_i\right)\left(\bar{v}_{i+n} - v_{i+p}\right). \tag{3.2}$$

Note that this equation in the special case $p = n+1$ appeared in [8]. One can check, that it can be also written as

$$\left(v_{i+p} - v_i\right)\left(\bar{v}_{i+p} - v_{i+p-n}\right) = \left(\bar{v}_{i+p} - \bar{v}_i\right)\left(\bar{v}_{i+n} - v_i\right) \tag{3.3}$$

and in the form

$$\left(\bar{v}_{i+n} - v_i\right)\left(\bar{v}_i - v_{i+p-n}\right) = \left(\bar{v}_{i+p} - v_{i+p-n}\right)\left(\bar{v}_{i+n} - v_{i+p}\right). \tag{3.4}$$

We observe that replacing $v_i \leftrightarrow \bar{v}_i$ and $n \leftrightarrow p-n$ in (3.2), we obtain (3.3). This means that two different pairs of parameters (p, n) and $(p, p-n)$ correspond in fact to the same equation (3.2).

Making use of (3.4), we observe that this quadratic equation has the following integral:

$$I_i = \prod_{j=1}^{n} g_{i+j-1} \cdot \prod_{j=1}^{p-n} h_{i+j-1}, \tag{3.5}$$

where $h_i \equiv g_{i+n} + s_i = \bar{v}_{i+n} - v_i$. Remark, that we can also present I_i in the form

$$I_i = \prod_{j=1}^{p} \left(\bar{v}_{i+j-1} - v_{i+j+p-n-1}\right).$$

The latter needs some explanation. This formula involve $\bar{v}_{i+\alpha}$ with $\alpha \in \{0, \ldots, p-1\}$, while α in $v_{i+\alpha}$ is calculated modulo p.

3.2. **Some simple examples.** Remark that the equation $I_i = c$ with some constant c in simplest case $n = 1$ and $p = 2$, namely,

$$(\bar{v}_i - v_{i+1})(\bar{v}_{i+1} - v_i) = c \tag{3.6}$$

is nothing else but lattice potential KdV (lpKdV) equation [5, 6] also known as H_1 equation in Adler-Bobenko-Suris classification [1]. Therefore one can consider the relation $I_i = c$ with I_i given by (3.5) in a sense as a generalization of the lpKdV equation. For example, in the case $p = 3$ we have two equations

$$(\bar{v}_i - v_{i+2})(\bar{v}_{i+1} - v_i)(\bar{v}_{i+2} - v_{i+1}) = c$$

for $n = 1$ and

$$(\bar{v}_i - v_{i+1})(\bar{v}_{i+1} - v_{i+2})(\bar{v}_{i+2} - v_i) = c$$

for $n = 2$. One sees that these two equations are in fact the same one.

3.3. **Discrete zero-curvature representation for the equation (3.2).** Let $\phi_{k,i} \equiv \phi_{i+k-1}$ for $k = 1, \ldots, p$ and $\Phi_i \equiv (\phi_{1,i}, \ldots, \phi_{p,i})^T$. Then we can rewrite (3.1) in matrix form $\bar{\Phi}_i = V_i \Phi_i$ or more explicitly as

$$\bar{\phi}_{1,i} = g_i \phi_{1,i} + \phi_{p-n+1,i}, \ldots, \bar{\phi}_{n,i} = g_{i+n-1}\phi_{n,i} + \phi_{p,i},$$

$$\bar{\phi}_{n+1,i} = h_i \phi_{n+1,i} + \frac{1}{z}\phi_{1,i}, \ldots, \bar{\phi}_{p,i} = h_{i+p-n-1}\phi_{p,i} + \frac{1}{z}\phi_{p-n,i}.$$

Obviously, the second equation which complete discrete zero-curvature representation for (3.2) being of the form $\Phi_{i+1} = U_i \Phi_i$ is explicitly given by the equations

$$\phi_{k,i+1} = \phi_{k+1,i} \quad \text{for} \quad k = 1, \ldots, p-1$$

and

$$\phi_{p,i+1} = \frac{1}{z}\phi_{1,i} + (v_{i+p} - v_i)\phi_{n+1,i}.$$

Then, the discrete zero-curvature representation for quadratic equation (3.2) is given by matrix equation $V_{i+1}U_i = \bar{U}_i V_i$.

3.4. **An example. Zero-curvature representation for lpKdV.** Consider simplest case $n = 1$ and $p = 2$ for which we have following pair of auxiliary linear equations:

$$\left(\begin{array}{c}\phi_{1,i+1}\\ \phi_{2,i+1}\end{array}\right) = \left(\begin{array}{cc}0 & 1\\ \frac{1}{z} & v_{i+2} - v_i\end{array}\right)\left(\begin{array}{c}\phi_{1,i}\\ \phi_{2,i}\end{array}\right)$$

and

$$\left(\begin{array}{c}\bar{\phi}_{1,i}\\ \bar{\phi}_{2,i}\end{array}\right) = \left(\begin{array}{cc}\bar{v}_i - v_{i+1} & 1\\ \frac{1}{z} & \bar{v}_{i+1} - v_i\end{array}\right)\left(\begin{array}{c}\phi_{1,i}\\ \phi_{2,i}\end{array}\right).$$

Let $\varphi_{1,i} = \phi_{1,i}$ and $\varphi_{2,i} = (v_{i+1} - v_i)\,\phi_{1,i} - \phi_{2,i}$. In terms of these new wave functions we have

$$\left(\begin{array}{c}\varphi_{1,i+1}\\ \varphi_{2,i+1}\end{array}\right) = \left(\begin{array}{cc}v_{i+1} - v_i & -1\\ -(v_{i+1} - v_i)^2 - \frac{1}{z} & v_{i+1} - v_i\end{array}\right)\left(\begin{array}{c}\varphi_{1,i}\\ \varphi_{2,i}\end{array}\right) \tag{3.7}$$

and

$$\left(\begin{array}{c}\bar{\varphi}_{1,i}\\ \bar{\varphi}_{2,i}\end{array}\right) = \left(\begin{array}{cc}\bar{v}_i - v_i & -1\\ (\bar{v}_i - v_i)\,(\bar{v}_{i+1} - \bar{v}_i) + (v_i - \bar{v}_{i+1})\,(v_{i+1} - v_i) - \frac{1}{z} & \bar{v}_i - v_i\end{array}\right)\left(\begin{array}{c}\varphi_{1,i}\\ \varphi_{2,i}\end{array}\right).$$

If we make use of (3.6) we obtain

$$\left(\begin{array}{c}\bar{\varphi}_{1,i}\\ \bar{\varphi}_{2,i}\end{array}\right) = \left(\begin{array}{cc}\bar{v}_i - v_i & -1\\ -(\bar{v}_i - v_i)^2 + c - \frac{1}{z} & \bar{v}_i - v_i\end{array}\right)\left(\begin{array}{c}\varphi_{1,i}\\ \varphi_{2,i}\end{array}\right).$$

One can see that the latter is more like (3.7). Let $c = \alpha - \beta$ and $1/z = \alpha - \lambda$, where λ is a new "spectral" parameter. Then, as a result we obtain well-known symmetric zero-curvature representation defined by a pair of equations

$$\left(\begin{array}{c}\varphi_{1,i+1}\\ \varphi_{2,i+1}\end{array}\right) = \left(\begin{array}{cc}v_{i+1} - v_i & -1\\ -(v_{i+1} - v_i)^2 + \lambda - \alpha & v_{i+1} - v_i\end{array}\right)\left(\begin{array}{c}\varphi_{1,i}\\ \varphi_{2,i}\end{array}\right)$$

and

$$\left(\begin{array}{c}\bar{\varphi}_{1,i}\\ \bar{\varphi}_{2,i}\end{array}\right) = \left(\begin{array}{cc}\bar{v}_i - v_i & -1\\ -(\bar{v}_i - v_i)^2 + \lambda - \beta & \bar{v}_i - v_i\end{array}\right)\left(\begin{array}{c}\varphi_{1,i}\\ \varphi_{2,i}\end{array}\right).$$

4. Conclusion

In this note we have shown three classes of differential-difference equations (2.7), (2.9) and (2.13). Equations in these classes are defined by pairs of co-prime integers $n \geq 1$ and $p > n$. Only in the case $p = n + 1$ these equations are evolutionary ones. Here we do not consider the question of constructing of integrable hierarchies associated with auxiliary equation

$$z s_i \phi_{i+n} + \phi_i = z\phi_{i+p}$$

or

$$z\psi_{i+n} + r_{i+n}\psi_i = z\psi_{i+p}. \tag{4.1}$$

We only remember that integrable hierarchies associated with (4.1) were constructed in explicit form in [7]. These hierarchies are shown to be directly related to KP flows. Also we constructed explicit form of integrable hierarchy on the field $v = v_i$ in the particular

case $p = n + 1$ in [8]. We are going to study this question in more detail in subsequent publications.

References

[1] V.E. Adler , A.I. Bobenko, Yu. B. Suris.Classification of integrable equations on quad-graphs. The consistency approach. *Commun. Math. Phys.* **233** (2003) 513-543

[2] O.I. Bogoyavlenskii.Some constructions of integrable dynamical systems. *Math. USSR-Izv.* **31** 47-76

[3] Y. Itoh. An H-theorem for a system of competing species. *Proc. Japan Acad.* **51** (1975) 374-379

[4] K. Narita. Soliton solutions to extended Volterra equation. *J. Phys. Soc. Japan* **51** (1982) 1682-1685

[5] F.W. Nijhoff, G.R.W. Quispel, H.W. Capel. Direct linearization of nonlinear difference-difference equations. *Phys. Lett. A* **97** (1983) 125-128

[6] F.W. Nijhoff, H.W. Capel. The discrete Korteweg-de Vries equation. *Acta Appl. Math.* **39** (1995) 133-158

[7] A.K. Svinin. On some class of homogeneous polynomials and explicit form of integrable hierarchies of differential-difference equations. *J. Phys. A: Math. Theor.* **44** (2011) 165206 (16 pp)

[8] A.K. Svinin. On some integrable lattice related by the Miura-type transformation to the Itoh-Narita-Bogoyavlenskii lattice. *J. Phys. A: Math. Theor.* **44** (2011) 465210 (8 pp)

[9] R.I. Yamilov. 1983 Classification of discrete evolution equations. *Usp. Mat. Nauk* **38** (1983) 155-156 (in Russian)

Andrei K. Svinin, Institute for System Dynamics and Control Theory, Siberian Branch of Russian Academy of Sciences, P.O. Box 292, 664033 Irkutsk, Russia

E-mail address: `svinin@icc.ru`